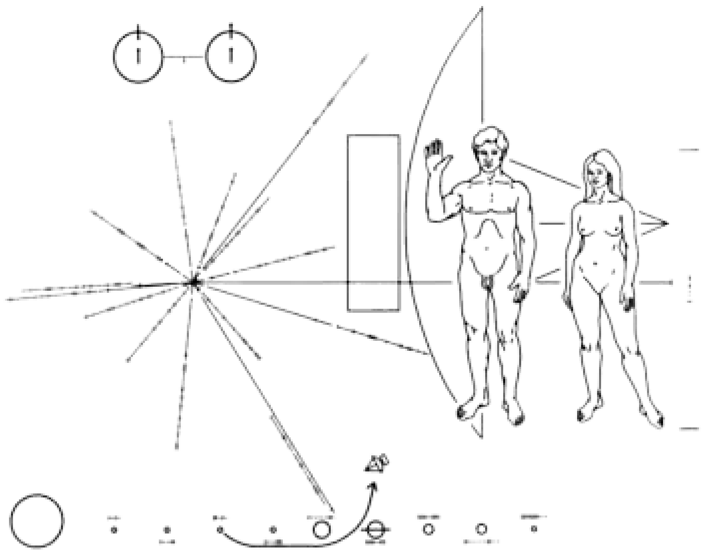
Mathematical language and mathematical progress

By Eberhard Knobloch

In 1972 the space probe Pioneer 10 was launched, carrying a plaque which contains the first message of mankind to leave our solar system[1]:



The space probe is represented by a circular segment and a rectangle designed on the same scale as that of the man and the woman. This is not true of the solar system: sun and planets are represented by small circles and points.

We do not know whether there are other intelligent beings living on stars in the universe, or whether they will even understand the message. But the non-verbal, the symbolical language of geometry seemed to be more appropriate than any other language. The grammar of any ordinary language is so complicated that still every computer breaks down with regards to it.

[1] Karl Märker, "Sind wir allein im Weltall? Kosmos und Leben", in: *Astronomie im Deutschen Museum, Planeten - Sterne - Welteninseln*, hrsg. von Gerhard Hartl, Karl Märker, Jürgen Teichmann, Gudrun Wolfschmidt (München: Deutsches Museum, 1993), p. 217. [We reproduce the image of the plaque from: http://en.wikipedia.org/wiki/Pioneer_plaque ; Karine Chemla]

The famous Nicholas Bourbaki wrote in 1948[2]: "It is the external form which the mathematician gives to his thought, the vehicle which makes it accessible to others, in short, the language suited to mathematics; this is all, no further significance should be attached to it". Bourbaki added: "To lay down the rules of this language, to set up its vocabulary and to clarify its syntax, all that is indeed extremely useful." But it was only the least interesting aspect of the axiomatic method for him.

Bourbaki emphasized that the vehicle or the language must not be mistaken for the content. Already Felix Klein and David Hilbert rejected the idea that mathematics could be reduced to a game based on a system of arbitrary rules and symbols. Bourbaki again underlined that the mathematician does not work like a machine, nor as the workman at a conveyor belt, but needs special intuition or rather a kind of direct divination (ahead of all reasoning) of the normal behaviour of mathematical beings.

There are several components of mathematical practice which hinder or favour the growth of mathematical knowledge, that is mathematical progress, for example, metamathematical views, proof standards, accepted questions, reasonings, statements, language to use Kitcher's expressions[3]. I would like to discuss the correlation between mathematical language and mathematical progress. Hence, I would like to discuss the following three aspects.

1. The language of calculi
   Capacity
   Drawbacks

2. Translations
   Entanglements
   Reductions
   Interpretations

3. Generalizations or the extension of language

[2] Bourbaki, "L'architecture des mathématiques", *Le Lionnais* 1948, pp. 35-47. I cite the English translation by Arnold Dresden: "The architecture of mathematics", *American Mathematical Monthly*, 57 (1950), pp. 221-232, reprinted in William Ewald, *From Kant to Hilbert*: *A source book in the foundations of mathematics*, vol. II (Oxford: Clarendon Press, 1996), pp. 1265-1276, here p. 1268.
[3] Philip Kitcher, *The nature of mathematical knowledge* (Oxford: University Press, 1983), p. 163.

New objects
New operations
New meanings

1. The language of calculi

Capacity

On 23 June 1798 the German romantic poet Novalis wrote in his "Mathematical notebook": "The number system is the model of a system of true language signs (eines wahren Sprachzeichensystems). Our letters are to become numbers, our language arithmetic".[4]

For Novalis the arithmetical calculus was the perfect model of a mathematical language. It disposes of peculiar symbols (the Indian-Arabic numerals), assigned meanings (the decimal numbers), rules of operation (the calculation techniques). The vocabulary provides the semantic dimension of language, the rules of operations are the grammar providing the syntactic dimension. The pragmatic dimension is formalized by symbols which exclude any misunderstanding by a subjective use of the signs or of the language system. When in 1929 Brouwer criticized the false belief in the magical character of the language of the Formalist School (as he called it) in his article "Mathematics, Science, and Language,"[5] he did not speak about arithmetic, but about pure mathematics in general. He maintained that even for pure mathematics there is no certain language which in the exchange of thought excludes misunderstanding and is free from error.

In the same year 1798 Etienne Bonnot de Condillac's book "The Language of the calculi" was posthumously published. Its first sentence reads as follows[6]: "Every language is an analytical method, and every analytical method is a language". Hence, the progress of analysis and the progress of language must be inter-

[4] Novalis, "Mathematischer Heft", in: Novalis, *Schriften*, vol. III, hrsg. von Richard Samuel in Zusammenarbeit mit Hans-Joachim Mähl und Gerhard Schulz (Stuttgart: W. Kohlhammer, 1968), p. 50.
[5] Luitzen Egbertus Jan Brouwer, "Mathematik, Wissenschaft und Sprache", *Monatshefte für Mathematik und Physik*, 36 (1929), pp. 153-164. I cite the English translation in Paolo Mancuso (ed.), *From Brouwer to Hilbert, The debate on the foundations of mathematics in the 1920s* (New York - Oxford: Oxford University Press, 1998), pp. 45-53, here p. 48.
[6] Etienne Bonnot de Condillac, *La langue des calculs* (Paris: Ch. Houel, 1798). I cite the German edition E. B. de Condillac, *Die Logik oder die Anfänge der Kunst des Denkens. Die Sprache des Rechnens*, hrsg. v. Georg Klaus, übers. von Erich Salewski. (Berlin: Akademie-Verlag, 1959), pp. 117-245, here p. 119.

dependent developments in human thought[7]. Mankind needs an artifical and symbolic language, in order to analyze the cluster of ideas into its constituent parts. Grammar reflects the steps in the evolution of the analytic method. The mathematical language Condillac used was algebra, though he admitted that mathematicians did not realize that algebra is a language in a true sense, since it had no grammar. He himself wanted to provide algebra with one. All languages ought to approximate the language of algebra, and insofar as they succeed, they will resemble one another in the perfection of their clarity[8].

We might ask why Condillac did not recommend Galileo's language of geometry. There cannot be any doubt about it: Because geometry disposes, it is true, of peculiar symbols and meanings, but not of rules of operations, as Augustus de Morgan[9] will explain in 1849: "No science of calculation can proceed without rules; and these geometry does not possess".

Condillac's own mathematical knowledge was rather limited. He relied on Euler's algebra or more difficult algebraic issues, or possible imperfections of the language of algebra. When in 1837 Hamilton published his "Essay on algebra as the science of pure time", he distingished between three different schools studying algebra according to which algebra is considered an instrument, or a language, or a contemplation[10]. The philological algebraist, it is true, might complain of imperfection, when his language presents him with an anomaly, when he finds an exception disturb the simplicity of his notation, or the symmetrical structure of his syntax; when a formula must be written with precaution, and a symbolism is not universal. But "no man can be so merely practical", he said, "as to use frequently the rules of Algebra, and never to admire the beauty of the language which expresses those rules, nor care to know the reasoning which deduces them."

Condillac praised the language of calculi, having only in mind common algebra, which was and remained indeed the model of a calculus: This is, de Morgan put it in 1849, "a science of calculation which has organized processes by which passage is

[7] Isabel F. Knight, *The geometric spirit. The Abbé de Condillac and the French enlightenment* (New Haven - London: Yale University Press, 1968), p. 166.
[8] Knight (1968), p. 175.
[9] Augustus de Morgan, *Trigonometry and double algebra* (London: Walton and Maberly, 1849). I cite the partial reprint in Ewald (1996), vol. I, pp. 350-361, here p. 351.
[10] William Rowan Hamilton, "Theory of conjugate functions, or algebraic couples: with a preliminary and elementary essay on algebra as the science of pure time", *Transactions of the Royal Irish Academy*, 17 (1837), pp. 293-422. I cite the partial reprint in Ewald (1996), vol. I, p. 369.

made, or may be made, mechanically from one result to another. A calculus always contains something which it would be possible to do by machinery"[11].

Hence it it clear right from the beginning that not every mathematical language is a calculus. A calculus displays two characteristics:

1) a system of symbolical notations

2) an algorithm of operations.

The first characteristic necessitates specification and precision. It does not imply, but enables certainty, mathematical reliability. The second characteristic simplifies the application of the calculus. It enables its fruitfulness and increases mathematical progress.

The language of ancient geometry had neither the one nor the other at its disposal. The language of proportions, which was the language of mathematical natural philosophy before the invention of Newton's and Leibniz's calculus, had not available the second characteristic: I remind of Kepler's formulation of his third planetary law or Galileo's formulation of the law of falling bodies. The language of indivisibles disposed of neither the one nor the other. What is more, the key notion "indivisible" remained controversial. While one author like Galileo[12] spoke of non-quantities -- a real catastrophe for the science of quantities, that is mathematics -- others like Pascal or Roberval interpreted them as infinitesimals.

Cavalieri justified indivisibles by comparing them with the algebraist's handling of irrational quantities[13]:

> "However ineffable, absurd and unknown the roots of numbers might be, the algebraists calculate with them and get the required results. In the same way the aggregates of lines or indivisibles might be nameless, absurd and unknown with respect to their number. Their magnitude is nevertheless inclosed in well-defined limits".

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[11] de Morgan (1849), p. 352.

[12] Galileo Galilei, *Discorsi e dimostrazioni matematiche, intorno a due nuove scienze* (Leyden: Elsevier, 1638), Giornata prima.

[13] Antoni Malet, *From Indivisibles to Infinitesimals, Studies on Seventeenth-Century Mathematizations of Infinitely Small Quantities* (Barcelona: Universitat Autònoma de Barcelona, 1996), p. 16.

Interestingly Georg Cantor chose a similar path in order to explain, to justify his transfinite numbers: They are, as he said in 1887, themselves in a certain sense new irrationalities[14]. He drew on his version of formalism for this. Ignorance was justified by another ignorance. No wonder that such a justification of indivisibles was not well accepted.

When Leibniz began to elaborate his infinitesimal analysis, he redefined indivisibles as infinitely small quantities which are smaller than any given quantity. He rightly emphasized the fertility of his still geometrical approach. He called his geometrically deduced "transmutation theorem" by means of which he could obtain all results so far gained in the field of geometrical quadratures the most important and most universal, the fundamental theorem of geometry[15].

After the invention of this differential calculus, its notation and its algorithm, he re-evaluated all his former achievements in the domain of infinitesimal geometry. His calculus involved "fertility" and universality, that is the widest range of applicability. He did not need the above mentioned fundamental theorem any longer, because the calculus comprehended all that which could be deduced from it, hence also this theorem, which basically was nothing else than an integral transformation. He classified his calculus as a supplement of algebra[16]. It concerned the "geometry of measure", while the algebraical calculus concerned the "geometry of determination". Both calculi were understood as an analysis, as methods of discovery with regard to the geometry of magnitudes. His great success in this respect encouraged him to elaborate or to try to elaborate other calculi, since 1678 the geometrical calculus for the geometry of situation and the calculus of determinants, since 1679 a logical calculus.

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[14] Georg Cantor, Letter to Kurd Laßwitz, 15 February 1884. I cite the reprint in *Georg Cantor, Gesammelte Abhandlungen mathematischen und philosophischen Inhalts, mit erläuternden Anmerkungen sowie mit Ergänzungen aus dem Briefwechsel Cantor-Dedekind*, hrsg. von Ernst Zermelo. Nebst einem Lebenslauf Cantors von Adolf Fraenkel (Berlin: Julius Springer, 1932; reprint Berlin - Heidelberg - New York: Springer 1980), pp. 387-396, here p. 395.

[15] Gottfried Wilhelm Leibniz, *De quadratura arithmetica circuli ellipseos et hyperbolae cujus corollarium est trigonometria sine tabulis*, kritisch herausgegeben und kommentiert von Eberhard Knobloch (Göttingen: Vandenhoeck & Ruprecht 1993), p. 35f.

[16] Gottfried Wilhelm Leibniz, "De geometria recondita et analysi indivisibilium atque infinitorum", *Acta Eruditorum* July 1686, pp. 292-300. I cite the reprint in : G. W. Leibniz, *Mathematische Schriften*, hrsg. von C. I. Gerhardt, vol. V (Halle: H. W. Schmidt, 1858); reprint Hildesheim: Olms, 1962), pp. 226-233, here 232.

As so many other of his scientific achievements, his calculus of determinants remained unknown. Nearly 200 years later James Joseph Sylvester emphatically praised this other example of algorithmic thinking[17]:
"For what is the theory of determinants? It is an algebra upon algebra; a calculus which enables us to combine and foretell the results of algebraical operations, in the same way as algebra itself enables us to dispense with the performance of the special operations of arithmetic. All analysis must ultimately close itself under this form".

The algebraizised differential calculus implied a double liberation, the liberation from geometry and the liberation from imagination. The same is true of the calculus of variations. When in 1744 Euler published his famous book on this subject, he still entitled it "Method of finding curved lines which have a maximum or minimum property or solution of the isoperimetric problem in the broadest sense". He used a geometrical approach, but admitted somewhat later[18] that the natural treatment of the subject should be free from geometrical considerations. On August 12, 1755 young Lagrange communicated to him the wanted analytical solution, his method using the symbol δ to express a variation.[19] Already in this letter Lagrange spoke of a calculus. Euler coined the expression "calculus variationum", calculus of variations, at once adopting Lagrange's method.

Obviously, every calculus is a method, while not every method is a calculus. Since then, an immense variety of calculi have been created: the calculus of probabilities, the calculus of matrices, the calculus of classes (the monadic predicate calculus of first order) by Neumann, Bernays, Gödel, the barycentric calculus of Möbius, the D-calculus theory of differential operators, the calculus of functional equations, Boole's algebra, to mention only a few.

Drawbacks

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[17] Eberhard Knobloch, "From Gauß to Weierstraß: Determinant Theory and Its Historical Evaluations", in: *The Intersection of History and Mathematics,* ed. by Sasaki Chikara, Sugiuru Mitsuo, Joseph W. Dauben (Basel - Boston - Berlin: Birkhäuser, 1994), pp. 51-66, here p. 52.
[18] Leonhard Euler, "Elementa calculi variationum", *Novi commentarii academiae scientiarum Petropolitanae*, 10 (1764), 1766, pp. 51-93. I cite the reprint in: L. Euler, *Opera omnia*, vol. I, 25, ed. by Constantin Carathéodory (Zürich: Orell Füssli, 1952), pp. 141-176, here p. 142.
[19] Lagrange to Euler, 12 August 1755, in: Leonhard Euler, *Commercium Epistolicum,* vol. V, ed. by Adolf P. Jushkevich and René Taton (Basel: Birkhäuser, 1980), pp. 366-375).

In spite of all its advantages - fertility, certainty, conclusiveness, universality - there are essential drawbacks of calculi. The most important is the following: a calculus cannot reveal the limit of its own capacity, the range of its successful applicability. As a consequence, mathematicians were and are inclined to believe, that the imperfection of the calculus is the reason for unsuccessful trials to use it. The calculus leads them astray. Let us consider three famous examples:
(1) Many generations of mathematicians, among them Leibniz and Euler, were convinced that the algebraic calculus was imperfect because it could not algorithmically solve general equations of fifth and higher degree. Completely new insights into the grouptheoretical behaviour of the roots of an equation were necessary, before Ruffini tried to prove in 1799, that the solvability of the general equation is impossible in principle.

(2) The most famous problem of dynamics is the so-called three body problem. Since 1750 more than 800 articles have been published about this subject, which was first discussed by Newton[20]:
According to Newton's second law let three mass points attract each other mutually. Then there is always an attractive force between two of them which is directly proportional to the product of masses of the two points and inversely proportional to their distances. The points may move freely and are originally in an arbitrary state of motion. Find the further motions.

Euler solved some special cases of the general problem but complained about the imperfection of the differential and integral calculus, that hindered him to solve it[21]. In his "Celestial Mechanics" Laplace adhered to the same opinion, saying that the solution of problems of celestial mechanics depends on the exactness of the observations and on the perfection of analysis[22]. Only Poincaré demonstrated toward the end of the nineteenth century that the general problem is analytically unsolvable in the sense, that a complete, rigourous integration of the concerned differential equations is impossible[23].

[20] Isaac Newton, *Philosophiae naturalis principia mathematica* (London: Joseph Streater, 1687), Sectio XI De Motu Corporum Sphaericorum viribus centripetis se mutuo petentium.
[21] Leonhard Euler, "Considérations sur le problème des trois corps", *Mémoires de l'Adadémie des Sciences de Berlin,* 19 (1763), 1770, pp. 194-220. Vol. II, 26 of Euler's *Opera omnia* which will contain this article has not yet appeared.
[22] Renate Wahsner (Hrsg.), *Mensch und Kosmos - die copernicanische Wende* (Berlin: Akademie Verlag, 1978), p. 326.
[23] Henri Poincaré, *Les méthodes nouvelles de la mécanique céleste* (Paris: Gauthier-Villars, 1892), vol. I. Introduction.

(3) When Cantor elaborated his calculus of transfinite cardinal numbers, he designed the smallest of them by , the power of the set of natural numbers. By means of this arithmetic of cardinal numbers, published in the years 1895 to 1897 but known earlier probably, it can be demonstrated, that the power of the set of all subsets of is 2 = c, c being the power of the continuum, represented for example by (0,1)[24]. In 1878 Cantor mentioned the continuum problem for the first time: The question arose: Is it possible that there are other cardinalities between the power of denumerable sets and the power of the continuum of real numbers? Cantor presumed that this cannot be the case, that

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But he could not prove this continuum hypothesis: the calculus of transfinite cardinal numbers failed in this regard. Only in 1963 Paul Cohen found the unexpected answer: He demonstrated by a special proof method called "forcing", that the continuum hypothesis is undecidable in the axiomatic system ZFC, that is of Ernst Zermelo and Abraham Fraenkel, including the axiom of choice, if that is consistent.

To sum up: The language of calculi does not, cannot speak about itself. Furthermore, there is an overwhelming richness of mathematical questions which cannot be solved by means of it, for example, the search for the distribution of prime numbers, the emergence of non-Euclidean geometries, the inquiries into topological structures, graph-theoretical techniques etc.

2. Translations

In 1591, François Viète published his "Introduction into the analytic art". His new art was indeed the old forgotten art: his "new algebra" was the restored mathematical analysis of the ancients, their method of discovery. It is, as he said, the most certain inventor of the whole mathematics. It rightly claims to solve every problem:[25]

"nullum non problema solvere"

No wonder that such a science attracted the efforts of mathematicians, though some mathematicians like Isaac Newton's teacher Isaac Barrow contested that it was a science at all, for reasons, which we have to discuss a bit later. But while, for Barrow,

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24 Walter Purkert, Hans Joachim Ilgauds, *Georg Cantor, 1845-1918* (Basel - Boston - Stuttgart: Birkhäuser, 1987), p. 67.

25 François Viète, *In artem analyticen isagoge* (Tours: J. Mettayer, 1591). I cite the reprint in Fr. Viète, *Opera mathematica*, ed. by Frans van Schooten (Leiden: B. u. A. Elzevier, 1646; reprint Hildesheim - New York: Olms, 1970), pp. 1-12, here p. 12.

the algebraic, symbolical style was a system of abbreviations rather than an instrument of discovery[26], Descartes, Newton and many others took the opposite view. Let us consider, how they used the translation into mathematical languages in order to solve problems.

Entanglements

Descartes wanted to solve all geometrical problems, that meant for him all construction problems. His solution method consisted of two parts:

(1) He translated the geometrical problem into an algebraic equation, that is algebra provided the transfer of the problem to the equation.

(2) He reduced the equation to one of the standard formulae whose roots were geometrically constructed.

Hence he was one of the first important theorists of algebraic curves and functions. His method implied, it is true, some serious consequences:

1. A justification of such an algebraically interpreted geometry

It was unknown to the ancients. While modernists like John Wallis emphatically adhered to such algebraic techniques, John Hobbes rejected them just as much. Newton liked the heuristic expediency of the analytic methods, but he stood up for a separation of arithmetic from geometry and defended ancient synthetic geometry.

2. The restriction of the geometrical universe

Only those curves could be taken into account which could be described by an algebraic equation. Descartes called them 'geometric curves'. All other curves like the spiral or the logarithmic curve did not belong to his geometric universe; they were called 'mechanical'4. Hence the translation process defined which curve was geometrical and which was not: it restricted the geometrical universe.

3. A growing interest in algebraic techniques

It led to their liberation from their geometrical background. Leonhard Euler's theory of curves was completely algebraisized. Conic sections were introduced as curves of second order[27]. Analysis was no longer the application of algebra to geometry, but its

[26] Helena M. Pycior, *Symbols, impossible numbers, and geometric entanglements, British algebra through the commentaries on Newton's Universal Arithmetick* (Cambridge: University Press, 1997), p. 166.

[27] Leonhard Euler, *Introductio in analysin infinitorum* (Lausanne: M.-M. Bousquet, 1748).

own subject, the study of variables and functions. Lagrange underlined that his solutions could be understood without figures.

Reductions

For Newton arithmetic was so instrumental to algebra in all its operations, that they seemes jointly to constitute but a single, complete computing science[28] For Newton the chief differences between arithmetic and algebra concerned degrees of generality and methods[29]. "Whoever enters upon this science should in the first place understand the symbols and also learn the fundamental operations... Then let him practice these operations by reducing problems to equations"[30].This reduction consisted in symbolizing the meaning of the question in a language which is analytical[31]. The reduction process is called translation by Newton who explicitly said[32]: "For the solution of questions whose preoccupation is merely with numbers, or the abstract relationship of quantities, almost nothing else is required than that a translation be made from the particular verbal language in which the problem is propounded into one which is algebraic; that is, into characters which are fit to symbolize our concepts regarding the relationships of quantities."

Newton gave long explanations for the translation process. The passage bears witness to his interest in the structure of language and its capacity to convey meaning. In his youth he even wrote a draft "Of an universal language"[33]. Here he admited that sometimes it can happen that the language with which the circumstances of a question are expressed may appear unsuitable to be transmuted into an algebraic one (sermo algebraicus). By introducing a few changes and paying heed to the meaning of words rather than to their phonetic form, the change-over (versio) will be rendered an easy one.

Newton's algebraic sympathies were mixed. His "Principia" was written "in the dress of classical geometry" or "in the language of geometrical figures"[34],.

[28] Isaac Newton, *Lectures on Algebra during 1673-1683*, in: I. Newton, *The mathematical papers*, vol. V, ed. by D. T. Whiteside (Cambridge: University Press, 1972), pp. 54-517, here p. 57.
[29] Pycior (1997), p. 191.
[30] Newton (1972), p. 57.
[31] Newton (1972), p. 131.
[32] Newton (1972), p. 133; Pycior (1997), p. 185.
[33] Richard S. Westfall, *Never at rest, A biography of Isaac* Newton (Cambridge: University Press, 1980), p. 88.
[34] Pycior (1997), p. 204.

Interpretations

Translation does not only serve as a mean for problem solving, but also as a mean for finding new theorems, and that in a systematic way. I would like to consider the following three examples:

(1) The relations between plane and spherical trigonometry

There is one law of cosines of plane trigonometry

$$a^2 = b^2 + c^2 - 2bc \cos \text{à}$$

If we try to translate it into the language of spherical trigonometry, we notice at once, that there will be not only one translation, because the sides with spherical triangles are circular arcs, that is, they are measured by their angles. Without knowing the solution, we expect two laws. And indeed, there is the cosine law for sides

$$\cos a = \cos b \ \cos c + \sin b \ \sin c \ \cos \text{à}$$

and the cosine law for an angle of a spherical triangle:

$$\cos \text{à} = - \cos \text{á} \ \cos \gamma + \sin \text{á} \ \sin \gamma \ \cos a$$

(2) The duality principle in projective geometry

In 1822 Poncelet[35] added the points at infinity or improper points, and the line at infinity or improper line which contains all these improper points to the normal points and lines of plane geometry. In such a way he created a complete analogy between the fundamental notions of point and line, which are correlated by the relation of "incidence". Thanks to this duality principle, the research process becomes a translation process: Points are replaced by (improper) lines and vice versa, "to lie on" is replaced by "to pass through". We need only the principle of dual language, in order to deduce Brianchon's theorem of 1806 concerning the hexagon from Pascal's theorem of 1640.[36]

(3) The axiomatic method

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[35] Jean Victor Poncelet, *Traité des propriétés projectives des figures; ouvrage utile à ceux qui s'occupent des applications de la géométrie descriptive et d'opérations géométriques sur le terrain* (Paris: Bachelier, 1822).

[36] Eberhard Knobloch, "L'analogie et la pensée mathématique", in: *Mathématiques et philosophie de l'antiquité à l'âge classique*, édité par Roshdi-Rashed (Paris: Edition du Centre National de la Recherche Scientifique, 1991), pp. 217-237, here p. 232f.

The essential aim of this method is, as Bourbaki underlined[37], exactly that which logical formalism by itself cannot supply according to Bourbaki, namely, the profound intelligibility of mathematics. The method is based on special structures each of which carries with it its own language. The concrete example of a certain structure is an interpretation of the elements and abstract operations. This effects "a considerable economy of thought". As soon as a mathematician "has recognized, among the elements which he is studying, relations which satisfy the axioms of a known type, he has at its disposal immediately the entire arsenal of general theorems which belong to the structure of that type"[38].

3. Generalizations or the extension of language

Jacques Peletier, a 17th century French mathematician underlined the rigour of mathematics which cannot be found in other sciences. Its singularity is shown by its language, which must be distinguished from rhetoric language. Hence, he rejected polemics in mathematics, because the truth intrudes itself. As algebraist he praised algebra which discloses miracles.[39] Unfortunately mathematics evolved in another way for many reasons. One reason was language problems.

What to do for example when the language changed, when it was extended by generalization, as happened to the language of algebra? This was true of all three linguistic aspects, of its signs, of its notions, and of its rules. As its language evolved, many new questions arose by analogy with previous questions. The answers remained controversial for hundreds of years.

New objects

Already Cardano's "Great Art" or "Algebra" and Viète's "Analytical Art" involved new objects which were closely connected with new results, language, methodological justifications, and a changing relationship with geometry[40].

(1) New objects were the "imaginary numbers", provided that they were really numbers or quantities.

(2) A new result was the algorithmic solution of the general cubic equation. But it seemed to be paradoxical that in the so-called "irreducible case" the real solutions

[37] Bourbaki (1948), p. 1268.
[38] Bourbaki (1948), p. 1272.
[39] Peletier's textbook on algebra "L'algèbre" appeared in Lyon in 1554.
[40] Pycior (1997), p. 10.

could only be represented by means of imaginary numbers. Only in 1675 Leibniz demonstrated that the imaginary is unavoidable in this case.[41]

(3) The new language consisted in the symbolization of algebra created by Viète and Descartes. The new language enabled Viète to find the relation between the coefficients of an equation and its roots.

(4) The new methodological justification relied on the liberation of algebra from geometry. It led to a general concept of power, so that symbolism and conceptualization were intimately intertwined[42].

The legitimacy of negative and imaginary numbers in the extended universe of algebra was disputed. While William Oughtred, Thomas Harriot and John Wallis emphatically defended the symbolical style of algebra, its clarity, inventiveness, and generality, algebra was not yet a science for his contemporary Isaac Barrow. It was completely rejected by Thomas Hobbes, who adhered to to the geometry of the ancients. Further, Thomas Simson underlined the superiority of the geometrical over the algebraical analysis. Wallis developed algebra as an extension of arithmetic, in which symbols stood for quantities. He accepted negative and imaginary numbers as quantities represented by symbols. In the 18th century George Berkeley defined arithmetic and algebra as sciences of signs. Existence is attributed to signs only thought of. This properly applies to differentials, too. While Leibniz called them useful fictions, Sylvestre François Lacroix defended the way of speaking about infinite and infinitely small quantities. "This langugage", he said in his monumental 'Treatise on the differential and integral calculus',[43] "being very convenient by its brevity and actually exact, was the pretext of a great number of objections, because it seemed to attribute an actual existence to mathematical infinite, which is, strictly speaking, only a negative idea."An algebraist could calculate with the signs even when the ontological status of the designated thing was not clarified[44]. Euler saw in imaginary numbers only convenient notations void of real meaning[45] or as Krämer put it in an

[41] Gottfried Wilhelm Leibniz, "De resolutionibus aequationum cubicarum triradicalium. De radicibus realibus, quae interventu imaginariarum exprimuntur. Deque sexta quadam operatione arithmetica" (October 1675), in: G.W. Leibniz, *Sämtliche Schriften und Briefe*, vol. VII, 2, bearbeitet von Eberhard Knobloch, Walter S. Contro, unter Mitarbeit von Nora Gaedeke (Berlin: Akademie Verlag, 1996), pp. 678-700.

[42] Pycior (1997), p. 117.

[43] Sylvestre François Lacroix, *Traité du calcul différentiel et du calcul intégral*. 2[e] éd. (Paris: Courcier, 1810), vol. I, p. 18.

[44] Imre Toth, "Wissenschaft und Wissenschaftler im postmodernen Zeitalter", in: *Wie sieht und erfährt der Mensch seine Welt?* Vortragsreihe der Universität Regensburg, hrsg. von Hans Bungert (Regensburg: Buchverlag der Mittelbayerischen Zeitung, 1987), pp. 85-153, here p. 114.

[45] Toth (1987), p.115.

article just published. The ideal of a science which has completely turned to a calculus can only be attained if we disdain the knowledge of that what really exists[46].

For Newton û-1 remained a sign of impossibility. For his pupil, MacLaurin, imaginary numbers were no real quantities, for Nicolas Saunderson impossible quantities[47].

These difficulties were a consequence of the conviction that symbols, the result of operations upon symbols, either meant quantity or nothing at all: mathematics was the science of quantities. Hence, 1 - 2 was called a quantity less than nothing, in defiance of the common notion that all conceivable quantities are greater than nothing. The square root of the negative quantity was an absurdity constructed upon an absurdity.

These historical experiences were the motivation for Augustus de Morgan in the 1840s to try to give a precise, purely syntactic description of a symbolic calculus, a symbolic algebra. He followed in the footsteps of George Peacock and Duncan Gregory who had developed a calculus of operations. De Morgan abandoned the meanings of symbols and at the same time those of the words which described them. He only retained peculiar symbols and rules of operations without assigned meanings. This symbolic algebra may become the grammar of a hundred distinct significant "algebras", as he put it in 1849[48]. This is an important aspect, because symbolic algebra is an art, not a science for him, and an apparently useless art, except as it may afterwards furnish the grammar of a science. The proficient in a symbolic calculus would naturally demand a supply of meaning, a remark which reminds us of Bourbarki's statement mentioned in the beginning. After deducing a certain number of rules, he raised the question whether a machine could be made to turn one of the allegedly equivalent combinations into the other[49] .

New operations

For Newton arithmetic was so instrumental to algebra and all its operations that they seemed jointly to constitute but a single, complete computing science[50] . Numbers

[46] Sibylle Krämer, "Kalküle als Repräsentation, Zur Genese des operativen Symbolismus in der Neuzeit", in: *Räume des Wissens, Repräsentation, Codierung, Spur*, hrsg. von Hans-Jörg Rheinberger, Michael Hagner, Bettina Wahrig-Schmidt (Berlin: Akademie Verlag, 1997), pp. 111-122, here p. 121.
[47] Pycior (1997), p. 200, 272, 290.
[48] de Morgan (1849), p. 358.
[49] de Morgan (1849), p. 361.
[50] Newton (1972), p. 57.

were replaced by general variables, the definite, particular approach by an indefinite, universal way ("indefinite et universaliter"). But the fundamental operations were retained. Hence he adopted the expression "Universal arithmetic" for algebra, which the editor of his Algebra William Whiston published in 1707.

Though Joseph-Louis Lagrange did not reject this expression in his "Treatise on the resolution of numerical equations of all degrees"[51] , he emphasized that the true difference between arithmetic and algebra consists in another aspect: the essential character of algebra consists in the fact, that the results of its operations do not give the individual values of quantities looked for, those of arithmetical operations or geometrical constructions, but only represent the arithmetical or geometrical operations which have to be carried out on the first given quantities in order to obtain the quantities looked for. The table of these operations represented by the algebraic characters is that which is called formula in algebra[52] .

For him a quantity depending on other quantities in such a way that it can be expressed by a formula which contains these quantities, was a function of them. Hence in the true sense of the word, algebra is the art of determining the unknowns by means of functions of known quantities. In this sense, algebra is, as he said, in his "Lessons on the calculus of functions"[53] the science of functions, resulting from the arithmetical operations, which are generalized and transported to letters.

Lagrange interpreted the generalization of arithmetic in terms of a new language, in the language of the theory of functions[54]

theory of functions (algebra)

algebra proper calculus of functions
(generalized arithme- (algebraic operation of the

[51] Joseph Louis Lagrange, *Traité de la résolution des équations numériques de tous les degrés. Avec des notes sur plusieurs points de la théorie des équations algébriques.* 2e édition (Paris: Courcier, 1808). I cite the reprint in: *Oeuvres de Lagrange,* publiées par les soins de Joseph-Alfred Serret. vol. VIII (Paris: Gauthier-Villars, 1879).
[52] Lagrange (1808), p. 14f.
[53] Joseph Louis Lagrange, *Leçons sur le calcul des fonctions*. 2e édition (Paris: Courcier, 1806). I cite the reprint in: *Oeuvres de Lagrange*, publiées par les soins de Joseph-Alfred Serret. vol. X (Paris: Gauthier-Villars, 1884), p. 10.
[54] Umberto Bottazzini, "Introduction", in: Augustin-Louis Cauchy, *Cours d'analyse de l'Ecole Royale Polytechnique, Première Partie Analyse algébrique,* ed. by Umberto Bottazzini (Bologna: Editrice Bologna, 1992), p. XXXIX.

tical operations) the expansion in a series)

His calculus of functions is nothing else than the infinitesimal or transcendental analysis, but reduced to a purely algebraic origin[55] : the functions considered here result from the algebraic operation of an expansion in series, after indetermined increases have been attributed to one or to several quantities of the function. It is one of the great advantages of this calculus of functions that it furnishes expressions, which are as simple and as intelligible as the algebraic expression of powers and roots:

$$f(x+i) = f(x) + i f'(x) + \quad f''(x) + \quad f'''(x) + ...$$

Lagrange's approach avoided Leibniz's infinitely small quantities, Euler's expressions "Zero divided by zero", which does not present any idea, as he added[56], hence also Newton's ratios of vanishing quantities, though this method was the algebraic translation of the fluxional method, as he said. The expressions f'(x), f''(x) etc. are called "derived functions". Differential and integral calculus are the calculi of these functions[57] . We can, as Lagrange said[58] , regard the derivation of functions as a new, more general operation of algebra, whose range is by far greater than rising to a power. The differential expressions dy , d y , ... are rigorously speaking, only symbols, denoting functions, which are different from the primitive function y, but derived from it according to certain laws[59] .

New meanings

Lagrange reduced the "Theory of analytic functions" to the algebraic analysis of finite quantities, as he said in the subtitle of the concerned textbook. But his main tool were infinite series which Newton and Leibniz had called "equations of infinitely many terms", thus generalizing the notion of equation: "Equation" did not mean any longer a finite algebraic expression. But did it express an acutal quantity?

The extension of the algebraic language involved even divergent series: This means in modern terms series which have no sum, because there is no numerical limit of the sequence of their partial sums. In Euler's opinion, the quarrels about divergent

[55] Lagrange (1806), p. 9.
[56] Lagrange (1806), p. 7.
[57] Joseph Louis Lagrange, *Théorie des fonctions analytiques contenant les principes du calcul différentiel.* 2[e] éd. (Paris: Courcier, 1813). I cite the reprint in *Oeuvres de Lagrange,* publiées par les soins de Joseph-Alfred Serret. vol. IX (Paris: Gauthier-Villars, 1881), p. 16.
[58] Lagrange (1806), p. 19.
[59] Lagrange (1813), p. 413.

series were only verbal and arose in relation to the meaning one attached to the word "sum". Hence, he introduced a new definition: The "sum" of an infinite series is the finite expression which originates the series[60]. For example:

$$1 \; = 1 - x + x^2 - x^3 + \ldots$$

1 is the sum of the alternating series $1 - x + x^2 \ldots$

If we substitute $x = 1$, we get the result

$$1 = 1 - 1 + 1 - 1 \ldots$$

It was accepted by Euler, though the series is divergent according to modern terminology.

Euler gave a modern solution. He replaced the question "What is?" by the question "How shall we define?"[61] Euler's approach was not accepted. For Bolzano the right side of the equation was a quantative notation or representation devoid of objective reference like û-1 [62]. His reason depended on his conception of quantities: Among quantities the associative and the commutative laws must be valid, that is the following equation must be valid:

$$(A + B) + C = A + (B + C) = (A + C) + B$$

This is apparently not the case, hence the alternating series is not the expression of an actual quantity.

When Bolzano wrote his "Paradoxes of the infinite" in the late 1840s, his conception of quantity was already obsolete: Hamilton's quaternions of 1843 represented a non-commutative algebraic system.[63] The same applied to functional equations.

According to Euler's new definition any infinite series must have a "determinate" value. The word "sum" retained its usual meaning only for convergent series. Euler tacitly assumed that the same series could not arise from different expressions which yielded different values. Though complex analysis became in his hands a powerful tool in working with divergent series, they were not accepted by the following generation of mathematicians. In 1821 Cauchy rejected the "generality of algebra", which was so praised by Wallis as we heard, in the name of "geometrical rigour" and

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[60] Eberhard Knobloch (1991), p. 226.

[61] Godfrey Harold Hardy, *Divergent series* (Oxford: Clarendon Press, 1949; reprint 1973), p. 6.

[62] Bernhard Bolzano, *Paradoxien des Unendlichen*, hrsg. von Fr. Prihonsky (Leipzig: Reclam, 1851). I cite the partial reprint of the English translation by Donald A. Steele in Ewald (1996), vol. I, pp. 250-292, here pp. 280-283.

[63] Ewald (1996), vol. I, p. 376.

especially denied again that divergent series have a sum. Abel rejected them emphatically.[64]

About 20 years later, Riemann, however, explicitly demanded their use. When he tried in 1847 to elaborate a general conception of integration and differentiation[65] , he founded the derivation of fractionary order like d y on divergent series. Adopting Lagrange's terminology, Riemann defined these derivations in such a way that differentiation became a special case of them.

As a consequence, he felt bound to justify the use of divergent series against the majority of mathematicians of his time, saying: "Provided that the coefficients obey a certain law (and this is always presupposed), we can precisely indicate every particular part. As a consequence, it is a magnitude of which every part is exactly bounded and therefore determined. The mechanism of adding the numbers does not suffice to find its determined value. But this is no reason not to apply the rules to them which are proved for numerical magnitudes, and not to consider the results thus obtained as correct"[66]. Riemann never published this early mathematical essay. His pragmatism would not have settled the question: "How do you handle divergent series?" A new "grammar" was needed, because contrary to his statement the old "grammar", the old rules of operations actually failed.

[64] Umberto Bottazzini, *The Higher Calculus: A History of Real and Complex Analysis from Euler to Weierstrass* (New York - Berlin - Heidelberg etc.: Springer 1986), p. 87f.

[65] Berhard Riemann, "Versuch einer allgemeinen Auffassung der Integration und Differentiation", in: Bernhard Riemann, *Gesammelte mathematische Werke, Wissenschaftlicher Nachlass und Nachträge, collected papers,* nach der Ausgabe von Heinrich Weber und Richard Dedekind neu hrsg. von Raghavan Narasimhan (Berlin - Heidelberg: Springer 1990), pp. 385-398; Stéphane Dugowson, "L'élaboration par Riemann d'une définition de la dérivation d'ordre non entier", *Revue d'histoire des mathématiques*, 3 (1997), pp. 49-97.

[66] Riemann (1990), p. 387.